From (d) and (e), $\Psi\delta\psi+\Phi\delta\phi+\Theta\delta\theta=-\delta V\ldots(f)$.

$$\therefore\quad \Psi=-\frac{dV}{d\psi},\ \Phi=-\frac{dV}{d\phi},\ \text{and}\ \Theta=-\frac{dV}{d\theta}.$$

Therefore the Lagrangian equations of motion may be written:

$$\frac{d}{dt}\left(dT\Big/\frac{d\psi}{dt}\right)=\frac{dT}{d\psi}-\frac{dV}{d\psi},\ \frac{d}{dt}\left(dT\Big/\frac{d\phi}{dt}\right)=\frac{dT}{d\phi}-\frac{dV}{d\phi},$$

$$\text{and}\ \frac{d}{dt}\left(dT\Big/\frac{d\theta}{dt}\right)=\frac{dT}{d\theta}-\frac{dV}{d\theta}.$$

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from the February Number].

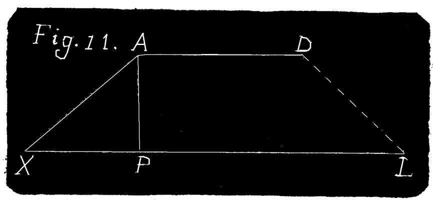
PPOPOSITION XIV. *The hypothesis of obtuse angle is inconsistent with Euclid's assumption: Two straight lines cannot enclose a space.*

Proof. From the hypthesis of obtuse angle, assumed as true, [and the first 28 propositions of Euclid], we have now deduced the truth of Euclid's Postulatum; that two straights will meet each other in some point toward those parts, toward which a certain straight, cutting them, makes two internal angles, of whatever kind, less than two right angles.

But this Postulatum holding good, on which Euclid supports himself after the twenty-eighth proposition of his first book, it is manifest to all Geometers, that the hypothesis of right angle alone is true, nor any place left for the hypothesis of obtuse angle. Therefore the hypothesis of obtuse angle is inconsistent with Euclid's assumption. Quod erat demonstrandum.

Otherwise, and more immediately.

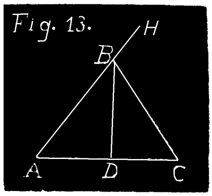
Since from the hypothesis of obtuse angle we have demonstrated (P. IX.) that two (fig. 11.) acute angles of the triangle APX, right-angled at P, are greater than one right angle; it follows that an acute angle PAD may be assumed such, that together with the aforesaid two acute angles it makes up two right angles. But then the straight AD must (by the preceding proposition,



joined to the hypothesis of obtuse angle) at length meet with this PL, or XL, regard being had to the secant, or incident AP; which is manifestly absurd against Eu. I. 17, if we regard the secant or incident AX.

PROPOSITION XV. *By any triangle ABC, of which the three angles (fig. 13.) are together equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle.*

Proof. For anyhow two angles of this triangle, as suppose A, and C, will be acute, because of Eu. I. 17. Wherefore the perpendicular, let fall from the apex of the remaining angle B upon this AC, will cut this AC (Eu. I. 17.) in some intermediate point D.



If therefore three angles of this triangle ABC are supposed equal to two right angles, it follows that all the angles of the triangles ADB, CDB will be together equal to four right angles, because of the two additional right angles at the point D. This holding good, now of neither of the said triangles, as suppose ADB, will the three angles together be less, or greater than two right angles; for thus viceversa the three angles together of the other triangle would be greater, or less than two right angles. Wherefore (P. IX.) from one triangle indeed would be established the hypothesis of acute angle, and from the other the hypothesis of obtuse angle; which is contrary to P. VI. and P. VII.

Therefore the three angles together of either of the aforesaid triangles will be equal to two right angles; and therefore (P. IX.) is established the hypothesis of right angle. Quod erat primo loco demonstrandum.

But if however the three angles of the proposed triangle ABC are taken greater than two right angles; now of the two triangles ADB, CDB all the angles together will be greater than four rignt angles, because of the two additional right angles at the point D.

This holding good; now of neither of the said triangles will the three angles together be precisely equal to, or less than two right angles; for thus viceversa the three angles of the other triangle would be together greater than two right angles. Wherefore (P.IX.) from one triangle indeed would be established the hypothesis either of right angle or of acute angle, and from the other the hypothesis of obtuse angle, which is contrary to P.V, P.VI, and P.VII.

Therefore the three angles together of either of the aforesaid triangles will be greater than two right angles; and therefote (P.IX.) is established the hypothesis of obtuse angle. Quod erat secundo loco demonstrandum.

But finally. If the three angles of the proposed triangle ABC are taken less than two right angles, now of the two triangles ADB, CDB, all the angles together will be less than four right angles, because of the two additional right angles at the point D.

This holding good; now of neither of the said triangles will the three angles together be equal to, or greater than two right angles; for thus viceversa of the other triangle the three angles together would be less than two right angles. Wherefore (P.IX.) from one triangle indeed would be establish-

ed the hypothesis either of right angle or obtuse angle, and from the other the hypothesis of acute angle; which is contrary to P.V, P.VI, and P.VII.

Therefore the three angles together of either of the aforesaid triangles will be less than two right angles; and therefore (P.IX.) is established the hypothesis of acute angle. Quod erat tertio loco demonstrandum.

Accordingly by any triangle ABC, of which the three angles are equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle. Quod erat propositum.

COROLLARY. Hence, any one side of any proposed triangle being produced, as suppose AB to H, the external angle HBC will be (Eu.I.13.) either equal to, or less, or greater than the remaining internal and opposite angles together at the points A and C, according as is true the hypothesis of right angle, or obtuse angle, or acute angle. And inversely.

[To be continued.]

THE "IRREDUCIBLE CASE."

By J. K. ELLWOOD, A. M., Principal Colfax School, Pittsburg, Pa.

PROBLEM.—*To extract the cube root of* $a \pm \sqrt{-b}$.

Put $\sqrt[3]{a+\sqrt{-b}}=m+n$, and $\sqrt[3]{a-\sqrt{-b}}=m-n$.

Then $a+\sqrt{-b}=m^3+3m^2n+3mn^2+n^3$, and $a-\sqrt{-b}=m^3-3m^2n+3mn^2-n^3$. Hence $a=m^3+3mn^2$, and $\sqrt{-b}=3m^2n+n^3$.

Example 1. Find the cube root of $9+25\sqrt{-2}$.

Here $a=m^3+3mn^2=9=3^3-18$. Hence $3mn^2=-18$, and $n=\sqrt{-2}$.

To verify these values of m and n substitute them in $\sqrt{-b}=3m^2n+n^3=25\sqrt{-2}$. Doing this we have $27\sqrt{-2}+(-2\sqrt{-2})=25\sqrt{-2}$.

$\therefore\ 3+\sqrt{-2}$ is the required root. When the substituted values of m and n do not give the second term they are not correct, and other values must be found by trial.

Example 2. Find the cube root of $2\sqrt{11}+30\sqrt{-3}$. Here $a=m^3+3mn^2=2\sqrt{11}=(\sqrt{11})^3-9\sqrt{11}$.

Hence $3mn^2=-9\sqrt{11}$, and $n=\sqrt{-3}$. Since these values of m and n substituted in $3m^2n+n^3$ give $\sqrt{-b}=30\sqrt{-3}$, the root is $\sqrt{11}+\sqrt{-3}$.

This method frequently enables us to simplify Cardan's formula for cubics in what is called the "irreducible case", said formula being

$$x=\sqrt[3]{q+\sqrt{(q^2-p^3)}}+\sqrt[3]{q-\sqrt{(q^2-p^3)}}.$$